

# MAXIMAL AND PRIMITIVE ELEMENTS IN BABY VERMA MODULES FOR TYPE $B_2$

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The purpose of this paper is to find maximal and primitive elements of baby Verma modules for a quantum group of type  $B_2$ . As a consequence the composition factors of the baby Verma modules are determined. Similar approach can be used to find maximal and primitive elements of Weyl modules for type  $B_2$ . In principle the results can be used to determine the module structure of a baby Verma module, but the calculations are rather involved, much more complicated than the case of type  $A_2$ .

For type  $A_2$ , submodule structure of a Weyl module has been determined in [DS1, I, K] and by Cline (unpublished). For type  $B_2$ , the socle series of Weyl modules was determined in [DS2]. In [X2] we determine the maximal and primitive elements in Weyl modules for type  $A_2$ , so that the Weyl modules are understood more explicitly. This paper is a sequent work of [X2], but less complete, since submodule structure of a baby Verma module is not determined. In this paper we only work with quantized enveloping algebras at roots of 1 (Lusztig version). For hyperalgebras the approach is completely similar, actually simpler.

The contents of the paper are as follows. In section 1 we recall some definitions and results about maximal and primitive elements. In section 2 we recall some facts about a quantized enveloping algebra of type  $B_2$ . In section 3 we determine the maximal and primitive elements in a Verma module of the (slightly enlarged) Frobenius kernel of type  $B_2$ . In section 4 we indicate that the maximal and primitive elements in a Weyl module for type  $B_2$  can

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be worked out similarly, but we omit the results. To avoid complicated expressions and for simplicity we assume that the order of the involved root of 1 is odd and greater than 3 and we only work with some special weights. The approach for general cases is completely similar.

## 1. MAXIMAL AND PRIMITIVE ELEMENTS

In this section we fix notation and recall the definition and some results for maximal and primitive elements. We refer to [L1-4, X1-2] for additional information.

**1.1.** Let  $U_\xi$  be a quantized enveloping algebra (over  $\mathbf{Q}(\xi)$ ) at a root  $\xi$  of 1 (Lusztig version). We assume that the rank of the associated Cartan matrix is  $n$  and the order of  $\xi \geq 3$ . As usual, the generators of  $U_\xi$  are denoted by  $e_i^{(a)}, f_i^{(a)}, k_i, k_i^{-1}$ , etc. Let  $\mathbf{u}$  be the Frobenius kernel and  $\tilde{\mathbf{u}}$  the subalgebra of  $U_\xi$  generated by all elements in  $\mathbf{u}$  and in the zero part of  $U_\xi$ . For  $\lambda \in \mathbf{Z}^n$  and a  $U_\xi$ -module (or  $\tilde{\mathbf{u}}$ -module  $M$ ) we denote by  $M_\lambda$  the  $\lambda$ -weight space of  $M$ . A nonzero element in  $M_\lambda$  will be called a vector of weight  $\lambda$  or a weight vector. Let  $m$  be a weight vector of a  $U_\xi$ -module (resp.  $\tilde{\mathbf{u}}$ -module)  $M$ . We call  $m$  **maximal** if  $e_i^{(a)}m = 0$  for all  $i$  and  $a \geq 1$  (resp.  $e_\alpha m = 0$  for all root vectors  $e_\alpha$  in the positive part of  $\tilde{\mathbf{u}}$ ). We call  $m$  a **primitive element** if there exist two submodules  $M_2 \subset M_1$  of  $M$  such that  $m \in M_1$  and the image in  $M_1/M_2$  of  $m$  is maximal. Obviously, maximal elements are primitive. We have (see [X2]):

(a) Let  $m \in M$  be a weight vector and let  $P_1$  be the submodule of  $M$  generated by  $m$ . Then  $m$  is primitive if and only if the image in  $P_1/P_2$  of  $m$  is maximal for some proper submodule  $P_2$  of  $P_1$ .

We shall write  $L(\lambda)$  (resp.  $\tilde{L}(\lambda)$ ) for an irreducible  $U_\xi$ -module (resp.  $\tilde{\mathbf{u}}$ -module) of highest weight  $\lambda$ .

(b) If  $m$  is a primitive element of weight  $\lambda$ , then  $L(\lambda)$  (or  $\tilde{L}(\lambda)$ ) is a composition factor of  $M$  (depending on  $M$  is a  $U_\xi$ -module or a  $\tilde{\mathbf{u}}$ -module).

(c) Let  $M$  and  $N$  be modules and  $\phi : M \rightarrow N$  a homomorphism. Let  $m$  be a weight vector in  $M$ . If  $\phi(m)$  is a primitive element of  $N$ , then  $m$  is a primitive element of  $M$ .

- (d) Let  $M, N, \phi, m$  be as in (c) and assume  $\phi(m) \neq 0$ . If  $m$  is a primitive element of  $M$ , then either  $\phi(m)$  is a primitive element of  $N$  or  $\phi(P_1) = \phi(P_2)$ , where  $P_1$  is the submodule of  $M$  generated by  $m$  and  $P_2$  is any submodule of  $P_1$  such that the image in  $P_1/P_2$  of  $m$  is maximal.
- (e) Let  $M, N, \phi, m$  be as in (c) and assume  $\phi(m) \neq 0$ . If  $m$  is a maximal element of  $M$ , then  $\phi(m)$  is a maximal element of  $N$ .

We shall denote by  $\tilde{Z}(\lambda)$  the (baby) Verma module of  $\tilde{\mathfrak{u}}$  with highest weight  $\lambda$  and denote by  $\tilde{1}_\lambda$  a nonzero element in  $\tilde{Z}(\lambda)_\lambda$ . Recall that to define  $U_\xi$  we need to choose  $d_i \in \{1, 2, 3\}$  such that  $(d_i a_{ij})$  is symmetric, where  $(a_{ij})$  is the concerned  $n \times n$  Cartan matrix. Let  $l_i$  be the order of  $\xi^{2d_i}$ . For  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$  we set  $\mathbf{l}\lambda = (l_1 \lambda_1, \dots, l_n \lambda_n)$ . We call  $\lambda$  is  $\mathbf{l}$ -restricted if  $0 \leq \lambda_i \leq l_i - 1$  for all  $i$ . Denote by  $\mathbf{N}_1^n$  the set of all  $\mathbf{l}$ -restricted elements in  $\mathbf{Z}^n$ . The following fact is well known.

- (f) Let  $\lambda \in \mathbf{N}_1^n$ ,  $\lambda' \in \mathbf{Z}^n$ . Set  $\mu = \lambda + \mathbf{l}\lambda' \in \mathbf{Z}^n$ . Then  $f_i^{(\lambda_i+1)} \tilde{1}_\mu$  is maximal in  $\tilde{Z}(\mu)$  if  $\lambda_i \neq l_i - 1$ .

## 2. SOME BASIC FACTS

**2.1.** From now on we assume that  $U_\xi$  is of type  $B_2$ . In this section we recall some basic facts about  $U_\xi$  and the Verma modules  $\tilde{Z}(\lambda)$ . For completeness and fix notations, we give the definition of  $U_\xi$  and  $\tilde{Z}(\lambda)$ .

Let  $a_{ii} = 2, a_{12} = -2, a_{21} = -1$ . Let  $U$  be the associative algebra over  $\mathbf{Q}(v)$  ( $v$  an indeterminate) generated by  $e_i, f_i, k_i, k_i^{-1}$  ( $i = 1, 2$ ) with relations

$$\begin{aligned}
k_1 k_2 &= k_2 k_1, & k_i k_i^{-1} &= k_i^{-1} k_i = 1 \\
k_i e_j &= v^{ia_{ij}} e_j k_i, & k_i f_j &= v^{-ia_{ij}} f_j k_i, \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{v_i - v_i^{-1}} \\
e_1 e_2^2 - (v^2 + v^{-2}) e_2 e_1 e_2 + e_2^2 e_1 &= 0 \\
e_1^3 e_2 - (v^2 + 1 + v^{-2}) e_1^2 e_2 e_1 + (v^2 + 1 + v^{-2}) e_1 e_2 e_1^2 - e_2 e_1^3 &= 0 \\
f_1 f_2^2 - (v^2 + v^{-2}) f_2 f_1 f_2 + f_2^2 f_1 &= 0 \\
f_1^3 f_2 - (v^2 + 1 + v^{-2}) f_1^2 f_2 f_1 + (v^2 + 1 + v^{-2}) f_1 f_2 f_1^2 - f_2 f_1^3 &= 0
\end{aligned}$$

where  $v_1 = v$  and  $v_2 = v^2$ . Let  $U'$  be the  $A = \mathbf{Z}[v, v^{-1}]$ -subalgebra of  $U$  generated by all  $e_i^{(a)} = e_i^a / [a]_i!$ ,  $f_i^{(a)} = f_i^a / [a]_i!$ ,  $k_i, k_i^{-1}$ ,  $a \in \mathbf{N}$ ,  $i = 1, 2$ ,

where  $[a]_i! = \prod_{h=1}^a \frac{v^{ih}-v^{-ih}}{v^i-v^{-i}}$  if  $a \geq 1$  and  $[0]_i! = 1$ . Note that  $\begin{bmatrix} k_i, c \\ a \end{bmatrix} = \prod_{h=1}^a \frac{v_i^{c-h+1} k_i - v_i^{-c+h-1} k_i^{-1}}{v_i^h - v_i^{-h}}$  is in  $U'$  for all  $c \in \mathbf{Z}$ ,  $a \in \mathbf{N}$ . We understand that  $\begin{bmatrix} k_i, c \\ a \end{bmatrix} = 1$  if  $a = 0$ . Note that  $f'_{12}{}^{(a)} = (f_1 f_2 - v^2 f_2 f_1)^a / [a]!$  and  $f_{12}^{(a)} = (f_2 f_1 - v^2 f_1 f_2)^a / [a]!$  are in  $U'$  for all  $a \in \mathbf{N}$ . Also

$$f'_{112}{}^{(a)} = \frac{(f_1 f'_{12} - f'_{12} f_1)^a}{(v + v^{-1})^a [a]_2!} \quad \text{and} \quad f_{112}^{(a)} = \frac{(f_{12} f_1 - f_1 f_{12})^a}{(v + v^{-1})^a [a]_2!}$$

are in  $U'$  for all  $a \in \mathbf{N}$ . Regard  $\mathbf{Q}(\xi)$  as an  $A$ -algebra by specializing  $v$  to  $\xi$ . Then  $U_\xi = U' \otimes_A \mathbf{Q}(\xi)$ . See [L3].

For convenience, the images in  $U_\xi$  of  $e_i^{(a)}, f_i^{(a)}, f'_{12}{}^{(a)}, f_{12}^{(a)}, f'_{112}{}^{(a)}, f_{112}^{(a)}, k_i, k_i^{-1}, \begin{bmatrix} k_i, c \\ a \end{bmatrix}$  etc. will be denoted by the same notation respectively. Let  $l$  be the order of  $\xi$  and  $l_i$  be the order of  $\xi^{2i}$ . In  $U_\xi$  we have  $e_i^{l_i} = f_i^{l_i} = 0$ . **For simplicity in this paper we assume that  $l$  is odd.** Then  $l_1 = l_2 = l$ . The Frobenius kernel  $\mathbf{u}$  of  $U_\xi$  is the subalgebra of  $U_\xi$  generated by all  $e_i, f_i, k_i, k_i^{-1}$ ,  $i = 1, 2$ . Its negative part  $\mathbf{u}^-$  is generated by all  $f_i$ . Note that  $f'_{12}{}^{(a)}, f_{12}^{(a)}, f'_{112}{}^{(a)}, f_{112}^{(a)}$  are in  $\mathbf{u}^-$  if  $0 \leq a \leq l-1$ . The subalgebra  $\tilde{\mathbf{u}}$  of  $U_\xi$  is generated by all  $e_i, f_i, k_i, k_i^{-1}, \begin{bmatrix} k_i, c \\ a \end{bmatrix}$ ,  $i = 1, 2; c \in \mathbf{Z}, a \in \mathbf{N}$ .

For  $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}^2$ , we denote by  $\tilde{I}_\lambda$  the left ideal of  $U_\xi$  generated by all  $e_i^{(a)}$  ( $a > 0$ ),  $k_i - \xi^{i\lambda_i}, \begin{bmatrix} k_i, c \\ a \end{bmatrix} - \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{\xi^i}$ . (We denote by  $\begin{bmatrix} b \\ a \end{bmatrix}_{\xi^i}$  the value at  $\xi^i$  of  $\prod_{h=1}^a \frac{v^{b-h+1} - v^{b+h-1}}{v^h - v^{-h}}$  for any  $b \in \mathbf{Z}$  and  $a \in \mathbf{N}$ .) The Verma module  $Z(\lambda)$  of  $U_\xi$  is defined to be  $U_\xi / \tilde{I}_\lambda$ . Let  $\tilde{1}_\lambda$  be the image in  $Z(\lambda)$  of 1. The Verma module  $\tilde{Z}(\lambda)$  of  $\tilde{\mathbf{u}}$  is defined to be the  $\tilde{\mathbf{u}}$ -submodule of  $Z(\lambda)$  generated by  $\tilde{1}_\lambda$ . Given non-negative integers  $a$  and  $b$ , we set

$$x_{a,b} = f_1^{(a)} f_2^{(a+b)} f_1^{(a+2b)} f_2^{(b)} = f_2^{(b)} f_1^{(a+2b)} f_2^{(a+b)} f_1^{(a)}.$$

Recall that  $l$  is the order of  $\xi$ . The following result is a special case of [X1, 4.2 (ii)].

(a) Assume  $0 \leq a, b \leq l-1$ ,  $c, d \in \mathbf{Z}$ , and let  $\mu = (lc-1+a, ld-1+b)$ . Then the element  $x_{a,b}$  is in  $\mathbf{u}^-$  and  $x_{a,b} \tilde{1}_\mu$  is maximal in  $\tilde{Z}(\mu)$  and generates the unique irreducible submodule of  $\tilde{Z}(\mu)$ . The irreducible submodule is isomorphic to  $\tilde{L}(lc-1-a, ld-1-b)$ .

The argument for [X1, 4.4(iv)] also gives the following result.

(b) Keep the assumption and notations in (a). Let  $p, q, s, t \in \mathbf{N}$  such that  $x = f_1^{(a+2b-pl)} f_2^{(a+b-ql)} f_1^{(a)}$  and  $y = f_2^{(a+b-sl)} f_1^{(a+2b-tl)} f_2^{(b)}$  are nonzero elements, then  $E_i x \tilde{1}_\mu = E_i x \tilde{1}_\mu = 0$  for  $i = 1, 2$ . If  $x$  and  $y$  are further in  $\mathbf{u}^-$ , then  $x \tilde{1}_\mu$  and  $y \tilde{1}_\mu$  are maximal in  $\tilde{Z}(\mu)$ .

We shall need a few formulas, which are due to Lusztig (see [L3, L4]). In  $U_\xi$  we have

$$(c) f_i^{(a)} f_i^{(b)} = \begin{bmatrix} a+b \\ a \end{bmatrix}_{\xi^i} f_i^{(a+b)},$$

$$(d) f_{12}^{(i)} f_2^{(j)} = \xi^{2ij} f_2^{(j)} f_{12}^{(i)},$$

$$(e) f_{112}^{(i)} f_{12}^{(j)} = \xi^{2ij} f_{12}^{(j)} f_{112}^{(i)},$$

$$(f) f_1^{(i)} f_{112}^{(j)} = \xi^{2ij} f_{112}^{(j)} f_1^{(i)},$$

$$(g) f_2^{(i)} f_{112}^{(j)} = \sum_{\substack{r,s,t \in \mathbf{N} \\ r+s=j \\ s+t=i}} \xi^{-2rs-2st} \prod_{h=1}^s (\xi^{-4h+2} - 1) f_{112}^{(r)} f_{12}^{(2s)} f_2^{(t)},$$

$$(h) f_{12}^{(i)} f_1^{(j)} = \sum_{\substack{r,s,t \in \mathbf{N} \\ r+s=j \\ s+t=i}} \xi^{-rs-st+s} \prod_{h=1}^s (\xi^{-2h} + 1) f_1^{(r)} f_{112}^{(s)} f_{12}^{(t)},$$

$$(i) f_2^{(i)} f_1^{(j)} = \sum_{\substack{r,s,t,u \in \mathbf{N} \\ s+t+u=i \\ r+2s+t=j}} \xi^{2ru+2su+rt} f_1^{(r)} f_{112}^{(s)} f_{12}^{(t)} f_2^{(u)}.$$

$$(j) f_{112}^{(i)} f_2^{(j)} = \sum_{\substack{r,s,t \in \mathbf{N} \\ r+s=j \\ s+t=i}} \xi^{-2rs-2st} \prod_{h=1}^s (\xi^{-4h+2} - 1) f_2^{(r)} f_{12}^{(2s)} f_{112}^{(t)},$$

$$(k) f_1^{(i)} f_{12}^{(j)} = \sum_{\substack{r,s,t \in \mathbf{N} \\ r+s=j \\ s+t=i}} \xi^{-rs-st+s} \prod_{h=1}^s (\xi^{-2h} + 1) f_{12}^{(r)} f_{112}^{(s)} f_1^{(t)},$$

$$(l) f_1^{(i)} f_2^{(j)} = \sum_{\substack{r,s,t,u \in \mathbf{N} \\ r+s+t=j \\ s+2t+u=i}} \xi^{2ru+2rt+us} f_2^{(r)} f_{12}^{(s)} f_{112}^{(t)} f_1^{(u)}.$$

(m) Assume  $0 \leq a_0, b_0 \leq l-1$  and  $a_1, b_1 \in \mathbf{Z}$ . We have

$$\begin{bmatrix} a_0 + a_1 l \\ b_0 + b_1 l \end{bmatrix}_{\xi^i} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}_{\xi^i} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix},$$

where  $\binom{a_1}{b_1}$  is the ordinary binomial coefficient.

Using (i), (l) and (m) we get

- (n) If  $0 \leq a \leq l-1$ , then  $f_1^{(l)} f_2^{(a)} - f_2^{(a)} f_1^{(l)}$  and  $f_2^{(l)} f_1^{(a)} - f_1^{(a)} f_2^{(l)}$  are in  $\mathbf{u}^-$ .  
(o) Let  $0 \leq a, b, c \leq l-1$ . Then  $f_1^{(a)} f_2^{(b)} f_1^{(c)} = 0$  and  $f_1^{(a)} f_2^{(l+b)} f_1^{(c)}$  is in  $\mathbf{u}^-$ , if  $a+c-2b \geq l$ . Similarly  $f_2^{(a)} f_1^{(b)} f_2^{(c)} = 0$  and  $f_2^{(a)} f_1^{(l+b)} f_2^{(c)}$  is in  $\mathbf{u}^-$ , if  $a+c-b \geq l$ .

The assertions (n) and (o) will be frequently used in computations.

Let  $\alpha_1 = (2, -1), \alpha_2 = (-2, 2) \in \mathbf{Z}^2$ . The set of positive roots is  $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ . Let  $W$  be the Weyl group generated by the simple reflections  $s_i$  corresponding to  $\alpha_i$ . Assume that  $l \geq 5$ . Then  $\langle \rho, \beta^\vee \rangle < l$  for all  $\beta \in R^+$ , where  $\rho = (1, 1)$ . For  $\lambda, \mu \in \mathbf{Z}^2$ , we write that  $\lambda \leq \mu$  if  $\mu - \lambda = a\alpha_1 + b\alpha_2$  for some non-negative integers  $a, b$ .

We say that  $x \in \mathbf{u}^-$  is homogeneous (of degree  $\beta$ ) if there exists  $\beta \in \mathbf{Z}^2$  such that  $\begin{bmatrix} k_i, c \\ a \end{bmatrix} x = x \begin{bmatrix} k_i, c + \langle \beta, \alpha_i^\vee \rangle \\ a \end{bmatrix}$  and  $k_i x = \xi^{i\langle \beta, \alpha_i^\vee \rangle} x k_i$  for all  $c \in \mathbf{Z}$  and  $a \in \mathbf{N}$ .

**2.2.** The  $W$ -orbit of  $\lambda = (0, 0)$  (dot action) consists of the following 8 elements,

$$\lambda, \quad s_1 \cdot \lambda = \lambda - \alpha_1, \quad s_2 \cdot \lambda = \lambda - \alpha_2, \quad s_2 s_1 \cdot \lambda = \lambda - \alpha_1 - 2\alpha_2, \quad s_1 s_2 \cdot \lambda = \lambda - 3\alpha_1 - \alpha_2, \\ s_1 s_2 s_1 \cdot \lambda = \lambda - 4\alpha_1 - 2\alpha_2, \quad s_2 s_1 s_2 \cdot \lambda = \lambda - 3\alpha_1 - 3\alpha_2, \quad s_1 s_2 s_1 s_2 \cdot \lambda = \lambda - 4\alpha_1 - 3\alpha_2.$$

Let  $a, b$  be integers and  $\lambda = (la, lb)$ . Using 1.1 (e-f) and 2.1 (a-b) we get

- (1) The following elements are maximal in  $\tilde{Z}(\lambda)$ :

$$\tilde{1}_\lambda, \quad f_1 \tilde{1}_\lambda, \quad f_2 \tilde{1}_\lambda, \quad f_2^{(2)} f_1 \tilde{1}_\lambda, \quad f_1^{(3)} f_2 \tilde{1}_\lambda, \\ f_2^{(2)} f_1^{(3)} f_2 \tilde{1}_\lambda, \quad f_1^{(3)} f_2^{(2)} f_1 \tilde{1}_\lambda, \quad f_1 f_2^{(2)} f_1^{(3)} f_2 \tilde{1}_\lambda.$$

- (2) Let  $\mu = \lambda + (l-1)\alpha_1$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\tilde{1}_\mu, \quad f_1^{(l-1)} \tilde{1}_\mu, \quad f_2^{(2)} \tilde{1}_\mu, \quad f_2 f_1^{(l-1)} \tilde{1}_\mu, \quad f_1^{(3)} f_2^{(2)} \tilde{1}_\mu, \\ f_2 f_1^{(3)} f_2^{(2)} \tilde{1}_\mu, \quad f_1^{(3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_\mu, \\ f_2^{(2)} f_1^{(l+3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_\mu.$$

- (3) Let  $\mu = \lambda + (l-1)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\tilde{1}_\mu, \quad f_1^{(3)} \tilde{1}_\mu, \quad f_2^{(l-1)} \tilde{1}_\mu, \quad f_2^{(2)} f_1^{(3)} \tilde{1}_\mu, \quad f_1 f_2^{(l-1)} \tilde{1}_\mu, \\ f_1 f_2^{(2)} f_1^{(3)} \tilde{1}_\mu, \quad f_2^{(2)} f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_\mu, \quad f_1^{(3)} f_2^{(l+2)} f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_\mu.$$

(4) Let  $\mu = \lambda + (l-1)\alpha_1 + (l-2)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\begin{aligned} \tilde{1}_\mu, \quad & f_1^{(3)}\tilde{1}_\mu, \quad f_2^{(l-2)}\tilde{1}_\mu, \quad f_2f_1^{(3)}\tilde{1}_\mu, \\ & f_1^{(l-1)}f_2^{(l-2)}\tilde{1}_\mu, \quad f_2f_1^{(l-1)}f_2^{(l-2)}\tilde{1}_\mu. \\ & f_1^{(l-1)}f_2^{(l+1)}f_1^{(3)}\tilde{1}_\mu, \quad f_1^{(3)}f_2^{(l+1)}f_1^{(2l-1)}f_2^{(l-2)}\tilde{1}_\mu. \end{aligned}$$

(5) Let  $\mu = \lambda + (l-3)\alpha_1 + (l-1)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\begin{aligned} \tilde{1}_\mu, \quad & f_1^{(l-3)}\tilde{1}_\mu, \quad f_2^{(2)}\tilde{1}_\mu, \quad f_2^{(l-1)}f_1^{(l-3)}\tilde{1}_\mu, \\ & f_1f_2^{(2)}\tilde{1}_\mu, \quad f_1f_2^{(l-1)}f_1^{(l-3)}\tilde{1}_\mu, \\ & f_2^{(l-1)}f_1^{(l+1)}f_2^{(2)}\tilde{1}_\mu, \quad f_2^{(2)}f_1^{(l+1)}f_2^{(l-1)}f_1^{(l-3)}\tilde{1}_\mu. \end{aligned}$$

(6) Let  $\mu = \lambda + (l-4)\alpha_1 + (l-2)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\begin{aligned} \tilde{1}_\mu, \quad & f_1^{(l-3)}\tilde{1}_\mu, \quad f_2\tilde{1}_\mu, \quad f_2^{(l-2)}f_1^{(l-3)}\tilde{1}_\mu, \\ & f_1^{(l-1)}f_2\tilde{1}_\mu, \quad f_1^{(l-1)}f_2^{(l-2)}f_1^{(l-3)}\tilde{1}_\mu. \\ & f_2^{(l-2)}f_1^{(l-1)}f_2\tilde{1}_\mu, \quad f_2f_1^{(l-1)}f_2^{(l-2)}f_1^{(l-3)}\tilde{1}_\mu. \end{aligned}$$

(7) Let  $\mu = \lambda + (l-3)\alpha_1 + (l-3)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\begin{aligned} \tilde{1}_\mu, \quad & f_1\tilde{1}_\mu, \quad f_2^{(l-2)}\tilde{1}_\mu, \quad f_2^{(l-1)}f_1\tilde{1}_\mu, \\ & f_1^{(l-3)}f_2^{(l-2)}\tilde{1}_\mu, \quad f_1^{(l-3)}f_2^{(l-1)}f_1\tilde{1}_\mu. \\ & f_2^{(l-1)}f_1^{(2l-3)}f_2^{(l-2)}\tilde{1}_\mu, \quad f_2^{(l-2)}f_1^{(2l-3)}f_2^{(l-1)}f_1\tilde{1}_\mu. \end{aligned}$$

(8) Let  $\mu = \lambda + (l-4)\alpha_1 + (l-3)\alpha_2$ . The following elements are maximal in  $\tilde{Z}(\mu)$ :

$$\begin{aligned} \tilde{1}_\mu, \quad & f_1^{(l-1)}\tilde{1}_\mu, \quad f_2^{(l-1)}\tilde{1}_\mu, \quad f_2^{(l-2)}f_1^{(l-1)}\tilde{1}_\mu, \\ & f_1^{(l-3)}f_2^{(l-1)}\tilde{1}_\mu, \quad f_1^{(l-3)}f_2^{(l-2)}f_1^{(l-1)}\tilde{1}_\mu. \\ & f_2^{(l-2)}f_1^{(2l-3)}f_2^{(l-1)}\tilde{1}_\mu, \quad f_2^{(l-1)}f_1^{(3l-3)}f_2^{(2l-2)}f_1^{(l-1)}\tilde{1}_\mu. \end{aligned}$$

### 3. MAXIMAL AND PRIMITIVE ELEMENTS OF $\tilde{Z}(\lambda)$ FOR TYPE $B_2$

In this section we determine the maximal and primitive elements in  $\tilde{Z}(\lambda)$  (or equivalently in any highest weight module of  $\tilde{\mathfrak{u}}$ ). To avoid complicated expressions we only work with some weights in the  $W$ -orbit of  $(0,0)$ . For general cases the approach is completely similar. Throughout the paper  $l$  is odd and is greater than or equal to 5.

**Theorem 3.1.** *Let  $a, b$  be integers and  $\lambda = (la, lb)$ . Then*

(i) *The following 8 elements are maximal in  $\tilde{Z}(\lambda)$ :*

$$\begin{aligned} \tilde{1}_\lambda, \quad f_1 \tilde{1}_\lambda, \quad f_2 \tilde{1}_\lambda, \quad f_2^{(2)} f_1 \tilde{1}_\lambda, \quad f_1^{(3)} f_2 \tilde{1}_\lambda, \\ f_2^{(2)} f_1^{(3)} f_2 \tilde{1}_\lambda, \quad f_1^{(3)} f_2^{(2)} f_1 \tilde{1}_\lambda, \quad f_1 f_2^{(2)} f_1^{(3)} f_2 \tilde{1}_\lambda. \end{aligned}$$

(ii) *The following 12 elements are primitive elements in  $\tilde{Z}(\lambda)$  but not maximal:*

$$\begin{aligned} [f_1^{(3)}, f_2^{(l)}] f_2 \tilde{1}_\lambda, \quad \frac{x_{l-1,2}}{f_1^{(l-1)}} \tilde{1}_\lambda, \quad [f_2^{(2)}, f_1^{(l)}] f_1^{(3)} f_2 \tilde{1}_\lambda \\ [f_2^{(2)}, f_1^{(l)}] f_1 \tilde{1}_\lambda, \quad \frac{x_{3,l-1}}{f_2^{(l-1)}} \tilde{1}_\lambda, \quad \frac{x_{3,l-1}}{f_1^{(l)} f_2^{(l-1)}} \tilde{1}_\lambda, \quad [f_1^{(3)}, f_2^{(l)}] f_2^{(2)} f_1 \tilde{1}_\lambda, \\ \frac{x_{3,l-2}}{f_1^{(l-1)} f_2^{(l-2)}} \tilde{1}_\lambda, \quad f_2^{(l-1)} f_1^{(l)} f_2 \tilde{1}_\lambda, \quad f_1 f_2^{(l-1)} f_1^{(l)} f_2 \tilde{1}_\lambda, \\ \frac{x_{l-1,l-1}}{f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)}} f_2 \tilde{1}_\lambda, \quad \frac{x_{l-1,l-1}}{f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)}} f_1 \tilde{1}_\lambda. \end{aligned}$$

(See 1.1 for the definition of  $x_{a,b}$ . Convention:  $[x, y] = xy - yx$  and  $\frac{x}{y}$  stands for an arbitrary homogeneous element  $z$  in  $\mathfrak{u}^-$  such that  $zy = x$ .)

Moreover no maximal element in  $\tilde{Z}(\lambda)$  has the same weight as any of the above 12 elements.

(iii) *The maximal and primitive elements in (i-ii) provide 20 composition factors of  $\tilde{Z}(\lambda)$ , which are*

$$\begin{aligned} \tilde{L}(\lambda), \quad \tilde{L}(\lambda - \alpha_1), \quad \tilde{L}(\lambda - \alpha_2), \\ \tilde{L}(\lambda - \alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - 3\alpha_1 - \alpha_2), \\ \tilde{L}(\lambda - 3\alpha_1 - 3\alpha_2), \quad \tilde{L}(\lambda - 4\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - 4\alpha_1 - 3\alpha_2), \\ \tilde{L}(\lambda - 3\alpha_1 - (l+1)\alpha_2), \quad \tilde{L}(\lambda - (l+3)\alpha_1 - (l+3)\alpha_2), \quad \tilde{L}(\lambda - (l+3)\alpha_1 - 3\alpha_2), \\ \tilde{L}(\lambda - (l+1)\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - (2l+4)\alpha_1 - (l+2)\alpha_2), \quad \tilde{L}(\lambda - (l+4)\alpha_1 - (l+2)\alpha_2), \\ \tilde{L}(\lambda - 4\alpha_1 - (l+2)\alpha_2), \quad \tilde{L}(\lambda - (l+3)\alpha_1 - (l+1)\alpha_2), \quad \tilde{L}(\lambda - l\alpha_1 - l\alpha_2), \\ \tilde{L}(\lambda - (l+1)\alpha_1 - l\alpha_2), \quad \tilde{L}(\lambda - 2l\alpha_1 - 2l\alpha_2), \quad \tilde{L}(\lambda - 2l\alpha_1 - l\alpha_2). \end{aligned}$$



Moreover,  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

*Proof:* (i) According to 1.1 (e-f), we see that (i) is true.

(ii) Now we argue for (ii).

(1) Consider the homomorphism:

$$\varphi_1 : \tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l-1)\alpha_1), \quad \tilde{1}_\lambda \rightarrow t_1 = f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1}.$$

Let

$$x_1 = (f_1^{(3)} f_2^{(l)} - f_2^{(l)} f_1^{(3)}) f_2 \in \mathfrak{u}^-.$$

Note that

$$f_1^{(3)} f_2^{(l+1)} t_1 = f_1^{(3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1} = x_1 t_1.$$

Using 1.1 (c) we see that  $x_1 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{38} = \lambda - 3\alpha_1 - (l+1)\alpha_2$ .

Let

$$y_1 = (f_2^{(2)} f_1^{(l)} - f_1^{(l)} f_2^{(2)}) x_1 \in \mathfrak{u}^-.$$

Note that  $f_2^{(2)} f_1^{(3)} f_2^{(l+1)} f_1^{(l-1)} = 0$ . We then can check that

$$f_2^{(2)} f_1^{(l+3)} f_2^{(l+1)} t_1 = f_2^{(2)} f_1^{(l+3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1} = y_1 t_1.$$

Using 1.1 (c) we see that  $y_1 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{48} = \lambda - (l+3)\alpha_1 - (l+3)\alpha_2$ . Note that we have  $y_1 = \frac{x_{l-1,2}}{f_1^{(l-1)}}$ .

Note that  $f_2^{(l+1)} f_1^{(l-1)} = f_2 f_2^{(l)} f_1^{(l-1)}$ , so we have  $x f_2^{(l+1)} f_1^{(l-1)} = 0$  if  $x f_2 = 0$  and  $x \in \mathfrak{u}^-$ . Thus we have a homomorphism (recall that  $\tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2}$  is also an element of the Verma module  $Z(\lambda + (l-1)\alpha_1 + l\alpha_2)$  of  $U_\xi$ ):

$$\psi_1 : \tilde{\mathfrak{u}} f_2 \tilde{1}_\lambda \rightarrow \tilde{\mathfrak{u}} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2},$$

$$f_2 \tilde{1}_\lambda \rightarrow f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2}.$$

Note that  $\psi_1(f_1^{(3)} f_2 \tilde{1}_\lambda) = f_1^{(3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2}$  and  $f_1^{(3)} f_2^{(l+1)} f_1^{(l-1)}$  is in  $\mathfrak{u}^-$ .

Let  $z_1 = f_2^{(2)} f_1^{(l)} - f_1^{(l)} f_2^{(2)} \in \mathfrak{u}^-$ . Using 1.1 (c) we see that  $z_1 f_1^{(3)} f_2 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{47} = \lambda - (l+3)\alpha_1 - 3\alpha_2$ . Note that  $z_1 f_1^{(3)} f_2 = \frac{x_{l-1,2}}{f_2^{(l)} f_1^{(l-1)}}$ .

(2) Now we consider the homomorphism:

$$\varphi_2 : \tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l-1)\alpha_2), \quad \tilde{1}_\lambda \rightarrow t_2 = f_2^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_2}.$$

Let

$$x_2 = (f_2^{(2)} f_1^{(l)} - f_1^{(l)} f_2^{(2)}) f_1 \in \mathfrak{u}^-.$$

Note that

$$f_2^{(2)} f_1^{(l+1)} t_2 = f_2^{(2)} f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_2} = x_2 t_2.$$

Using 1.1 (c) we see that  $x_2 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{31} = \lambda - (l+1)\alpha_1 - 2\alpha_2$ .

Let  $y_2$  be homogeneous in  $\mathfrak{u}^-$  such that  $y_2 t_2 = f_1^{(3)} f_2^{(l+2)} f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_\mu$ , here  $\mu = \lambda + (l-1)\alpha_2$ . According to 1.1 (c) we know that  $y_2 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{41} = \lambda - (2l+4)\alpha_1 - (l+2)\alpha_2$ . Note that  $y_2 = \frac{x_{3,l-1}}{f_2^{(l-1)}}$ .

As the reason for  $\psi_1$ , we have a homomorphism:

$$\psi_2 : \tilde{\mathfrak{u}} f_1 \tilde{1}_\lambda \rightarrow \tilde{\mathfrak{u}} f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+l\alpha_1+(l-1)\alpha_2},$$

$$f_1 \tilde{1}_\lambda \rightarrow f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+l\alpha_1+(l-1)\alpha_2}.$$

Note that  $\psi_2(f_2^{(2)} f_1 \tilde{1}_\lambda) = t'_2 = f_2^{(2)} f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+l\alpha_1+(l-1)\alpha_2}$  and  $f_2^{(2)} f_1^{(l+1)} f_2^{(l-1)}$  is in  $\mathfrak{u}^-$ . Let  $z_2$  be homogeneous in  $\mathfrak{u}^-$  such that

$$z_2 t'_2 = f_1^{(3)} f_2^{(l+2)} f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+l\alpha_1+(l-1)\alpha_2}.$$

Using 1.1 (c) we see that  $z_2 f_2^{(2)} f_1 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{42} = \lambda - (l+4)\alpha_1 - (l+2)\alpha_2$ . Note that  $z_2 f_2^{(2)} f_1 = \frac{x_{3,l-1}}{f_1^{(l)} f_2^{(l-1)}}$ .

We also have a homomorphism (recall that  $\tilde{1}_{\lambda+2l\alpha_1+(l-1)\alpha_2}$  is also an element of the Verma module  $Z(\lambda + 2l\alpha_1 + (l-1)\alpha_2)$  of  $U_\xi$ ) :

$$\theta_2 : \tilde{\mathfrak{u}} f_1 \tilde{1}_\lambda \rightarrow \tilde{\mathfrak{u}} f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+2l\alpha_1+(l-1)\alpha_2},$$

$$f_1 \tilde{1}_\lambda \rightarrow t''_2 = f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+2l\alpha_1+(l-1)\alpha_2}.$$

Let  $w_2 = [f_1^{(3)}, f_2^{(l)}] \in \mathfrak{u}^-$ . Then

$$w_2 f_2^{(2)} t''_2 = f_1^{(3)} f_2^{(l+2)} f_1^{(2l+1)} f_2^{(l-1)} \tilde{1}_{\lambda+2l\alpha_1+(l-1)\alpha_2}.$$

Using 1.1 (c) we see that  $w_2 f_2^{(2)} f_1 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{44} = \lambda - 4\alpha_1 - (l+2)\alpha_2$ . Note that  $w_2 f_2^{(2)} f_1 = \frac{x_{3,l-1}}{f_1^{(2l)} f_2^{(l-1)}}$ .

(3) Now we consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l-1)\alpha_1 + (l-2)\alpha_2),$$

$$\tilde{1}_\lambda \rightarrow t_3 = f_1^{(l-1)} f_2^{(l-2)} \tilde{1}_{\lambda+(l-1)\alpha_1+(l-2)\alpha_2}.$$

Let  $x_3$  be homogeneous in  $\mathbf{u}^-$  such that

$$x_3 t_3 = f_1^{(3)} f_2^{(l+1)} f_1^{(2l-1)} f_2^{(l-2)} \tilde{1}_{\lambda+(l-1)\alpha_1+(l-2)\alpha_2}.$$

Using 1.1 (c) we know that  $x_3 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{37} = \lambda - (l+3)\alpha_1 - (l+1)\alpha_2$ . Note that  $x_3 = \frac{x_{3,l-2}}{f_1^{(l-1)} f_2^{(l-2)}}$ .

(4) Since  $f_1^{(l-3)} f_2^{(l-2)} \in f_2 \mathbf{u}^-$  (see 2.1), we have a surjective homomorphism

$$\begin{aligned} \varphi_4 : \tilde{\mathbf{u}} f_2 \tilde{1}_\lambda &\rightarrow \tilde{\mathbf{u}} f_1^{(l-3)} f_2^{(l-2)} \tilde{1}_{\lambda+(l-3)\alpha_1+(l-3)\alpha_2} \\ f_2 \tilde{1}_\lambda &\rightarrow f_1^{(l-3)} f_2^{(l-2)} \tilde{1}_{\lambda+(l-3)\alpha_1+(l-3)\alpha_2}. \end{aligned}$$

Let  $x_4 = f_2^{(l-1)} f_1^{(l)} - f_1^{(l)} f_2^{(l-1)} \in \mathbf{u}^-$ . Then

$$x_4 f_2 \tilde{1}_\lambda = f_2^{(l-1)} f_1^{(l)} f_2 \tilde{1}_\lambda$$

is a primitive element of weight  $\gamma_{34} = \lambda - l\alpha_1 - l\alpha_2$ .

Let  $y_4 = f_1 x_4 f_2$ . Then  $y_4 \tilde{1}_\lambda$  is a primitive element of weight  $\gamma_{45} = \lambda - (l+1)\alpha_1 - l\alpha_2$ .

(5) Consider the homomorphism

$$\begin{aligned} \tilde{\mathbf{u}} f_2 \tilde{1}_\lambda &\rightarrow \tilde{\mathbf{u}} f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(l-3)\alpha_2} \\ f_2 \tilde{1}_\lambda &\rightarrow f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(l-3)\alpha_2}. \end{aligned}$$

Let  $x_5$  be homogeneous in  $\mathbf{u}^-$  such that

$$x_5 f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)} = f_2^{(l-1)} f_1^{(3l-3)} f_2^{(2l-2)} f_1^{(l-1)}.$$

By 1.1 (c),  $x_5 f_2 \tilde{1}_\lambda$  is primitive and is of weight  $\gamma_{35} = \lambda - 2l\alpha_1 - 2l\alpha_2$ . Note that  $x_5 = \frac{x_{l-1,l-1}}{f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)}}$ .

Consider the surjective homomorphism

$$\begin{aligned} \tilde{\mathbf{u}} f_1 \tilde{1}_\lambda &\rightarrow \tilde{\mathbf{u}} f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(2l-3)\alpha_2} \\ f_1 \tilde{1}_\lambda &\rightarrow f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(2l-3)\alpha_2}. \end{aligned}$$

Let  $y_5$  be homogeneous in  $\mathbf{u}^-$  such that

$$y_5 f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)} = f_2^{(l-1)} f_1^{(3l-3)} f_2^{(2l-2)} f_1^{(l-1)}.$$

By 1.1 (c),  $y_5 f_1 \tilde{1}_\lambda$  is primitive and is of weight  $\gamma_{32} = \lambda - 2l\alpha_1 - l\alpha_2$ . Note that

$$y_5 = \frac{x_{l-1,l-1}}{f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)}}.$$

We may also consider the homomorphism:

$$\begin{aligned} \tilde{\mathbf{u}} f_2 \tilde{1}_\lambda &\rightarrow \tilde{\mathbf{u}} f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(2l-2)\alpha_2}, \\ \tilde{\mathbf{u}} f_1 \tilde{1}_\lambda &\rightarrow f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(2l-4)\alpha_1+(2l-2)\alpha_2}. \end{aligned}$$

Using 1.1 (c) we know that  $[[f_2^{(l-1)}, f_1^{(l)}], f_1^{(l)}]f_2\tilde{1}_\lambda = \frac{x_{l-1,l-1}}{f_1^{(l-3)}f_2^{(2l-2)}f_1^{(l-1)}}f_2\tilde{1}_\lambda$  is also a primitive element of weight  $\lambda - 2l\alpha_1 - l\alpha_2$ .

The element  $f_1f_2^{(2)}f_1^{(3)}f_2\tilde{1}_\lambda$  generates the unique irreducible submodule of  $\tilde{Z}(\lambda)$ . Clearly, the weight of any element in (ii) is not greater than  $\lambda - 4\alpha_1 - 3\alpha_2$ , therefore no maximal element in  $\tilde{Z}(\lambda)$  has the same weight as any of the elements in (ii).

(iii) Using (i), (ii) and 1.1 (b), we see that  $\tilde{Z}(\lambda)$  has the 20 composition factors. The dimensions of irreducible  $\tilde{\mathbf{u}}$ -modules are known (see [APW]). By a comparison of dimensions we know that  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

The theorem is proved.

**Theorem 3.2.** *Let  $a, b$  be integers and  $\lambda = (la, lb - 3)$ . Then*

(i) *The following elements are maximal in  $\tilde{Z}(\lambda)$ :*

$$\begin{aligned} \tilde{1}_\lambda, \quad f_1\tilde{1}_\lambda, \quad f_2^{(l-2)}\tilde{1}_\lambda, \quad f_2^{(l-1)}f_1\tilde{1}_\lambda, \\ f_1^{(l-3)}f_2^{(l-2)}\tilde{1}_\lambda, \quad f_1^{(l-3)}f_2^{(l-1)}f_1\tilde{1}_\lambda. \\ f_2^{(l-1)}f_1^{(2l-3)}f_2^{(l-2)}\tilde{1}_\lambda, \quad f_2^{(l-2)}f_1^{(2l-3)}f_2^{(l-1)}f_1\tilde{1}_\lambda, \end{aligned}$$

$$\frac{f_2^{(l-1)}f_1^{(l-3)}}{f_2^{(2)}}\tilde{1}_\lambda, \quad \frac{f_1f_2^{(l-1)}f_1^{(l-3)}}{f_2^{(2)}}\tilde{1}_\lambda, \quad f_1^{(l-3)}[f_2^{(l-1)}, f_1^{(l)}]f_1\tilde{1}_\lambda.$$

(ii) *The following elements are primitive elements in  $\tilde{Z}(\lambda)$  but not maximal:*

$$\begin{aligned} \frac{x_{l-1,l-1}}{f_1^{(l-1)}}\tilde{1}_\lambda, \quad \frac{x_{l-1,l-1}}{f_2^{(l)}f_1^{(l-1)}}\tilde{1}_\lambda, \quad [f_2^{(l-1)}, f_1^{(l)}]f_1\tilde{1}_\lambda, \\ \frac{x_{3,l-1}}{f_2^{(2)}f_1^{(3)}}\tilde{1}_\lambda, \quad \frac{x_{3,l-1}}{f_2^{(l+2)}f_1^{(3)}}\tilde{1}_\lambda, \quad f_1^{(l-1)}f_2^{(l)}f_1\tilde{1}_\lambda, \\ \frac{x_{3,l-2}}{f_2f_1^{(3)}}f_1\tilde{1}_\lambda, \quad [f_2^{(l-2)}, f_1^{(l)}]\tilde{1}_\lambda, \quad \frac{x_{l-1,2}}{f_2f_1^{(3)}f_2^{(2)}}f_1\tilde{1}_\lambda. \end{aligned}$$

Moreover no maximal element in  $\tilde{Z}(\lambda)$  has the same weight as any of the above 9 elements.

(iii) *The maximal and primitive elements in (i-ii) provide 20 composition factors of  $\tilde{Z}(\lambda)$ , which are*

$$\begin{aligned} \tilde{L}(\lambda), \quad \tilde{L}(\lambda - \alpha_1), \quad \tilde{L}(\lambda - (l-2)\alpha_2), \\ \tilde{L}(\lambda - \alpha_1 - (l-1)\alpha_2), \quad \tilde{L}(\lambda - (l-3)\alpha_1 - (l-2)\alpha_2) \end{aligned}$$

$$\begin{aligned}
&\tilde{L}(\lambda - (l-2)\alpha_1 - (l-1)\alpha_2), & \tilde{L}(\lambda - (2l-3)\alpha_1 - (2l-3)\alpha_2), & \tilde{L}(\lambda - (2l-2)\alpha_1 - (2l-3)\alpha_2), \\
&\tilde{L}(\lambda - (l-3)\alpha_1 - (l-3)\alpha_2), & \tilde{L}(\lambda - (l-2)\alpha_1 - (l-3)\alpha_2), & \tilde{L}(\lambda - (3l-3)\alpha_1 - (3l-3)\alpha_2), \\
&\tilde{L}(\lambda - (3l-3)\alpha_1 - (2l-3)\alpha_2), & \tilde{L}(\lambda - (l+1)\alpha_1 - (l-1)\alpha_2), & \tilde{L}(\lambda - (2l-2)\alpha_1 - (l-1)\alpha_2), \\
&\tilde{L}(\lambda - (2l+1)\alpha_1 - (2l-1)\alpha_2), & \tilde{L}(\lambda - (2l+1)\alpha_1 - (l-1)\alpha_2), & \tilde{L}(\lambda - l\alpha_1 - l\alpha_2), \\
&\tilde{L}(\lambda - l\alpha_1 - (2l-2)\alpha_2), & \tilde{L}(\lambda - l\alpha_1 - (l-2)\alpha_2), & \tilde{L}(\lambda - 2l\alpha_1 - l\alpha_2).
\end{aligned}$$

Moreover,  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

*Proof:* (i) According to 1.1 (e-f), we see that the first 8 elements in (i) are maximal.

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + 2\alpha_2), \quad \tilde{1}_\lambda \rightarrow t_1 = f_2^{(2)} \tilde{1}_{\lambda+2\alpha_2}.$$

Since  $f_2^{(l-1)} f_1^{(l-3)}$  is in  $\mathbf{u}^- f_2^{(2)}$ , using 1.1 (c) we see that  $\frac{f_2^{(l-1)} f_1^{(l-3)}}{f_2^{(2)}} \tilde{1}_\lambda$  and  $\frac{f_1 f_2^{(l-1)} f_1^{(l-3)}}{f_2^{(2)}} \tilde{1}_\lambda$  are primitive elements of weights  $\lambda - (l-3)\alpha_1 - (l-3)\alpha_2$  and  $\lambda - (l-2)\alpha_1 - (l-3)\alpha_2$  respectively. One can check directly that the two elements are maximal. We will show that the last element in (i) is maximal in part (2) of the argument for (ii).

(ii) Now we argue for (ii).

(1) Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l-1)\alpha_1), \quad \tilde{1}_\lambda \rightarrow t_1 = f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1}.$$

Let  $x_1 = \frac{x_{l-1, l-1}}{f_1^{(l-1)}} \in \mathbf{u}^-$ . Using 1.1 (c) we see that  $x_1 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (3l-3)\alpha_1 - (3l-3)\alpha_2$ .

Consider the homomorphism:

$$\begin{aligned}
&\tilde{\mathbf{u}} f_2^{(l-2)} \tilde{1}_\lambda \rightarrow \tilde{\mathbf{u}} f_2^{(2l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2}, \\
&f_2^{(l-2)} \tilde{1}_\lambda \rightarrow f_2^{(2l-2)} f_1^{(l-1)} \tilde{1}_{\lambda+(l-1)\alpha_1+l\alpha_2}.
\end{aligned}$$

Let

$$y_1 = \frac{1}{2} [[f_2^{(l-1)}, f_1^{(l)}], f_1^{(l)}] f_1^{(l-3)}.$$

Then  $y_1 f_2^{(2l-2)} f_1^{(l-1)} = x_{l-1, l-1}$ . Using 1.1 (c) we see that  $y_1 f_2^{(l-2)} \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (3l-3)\alpha_1 - (2l-3)\alpha_2$ . Note that  $y_1 f_2^{(l-2)} = x_{l-1, l-1} / f_2^{(l)} f_1^{(l-1)} \in \mathbf{u}^-$ .

(2) Now we consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + 2\alpha_2), \quad \tilde{1}_\lambda \rightarrow t_2 = f_2^{(2)} \tilde{1}_{\lambda+2\alpha_2}.$$

Let

$$x_2 = (f_2^{(l-1)} f_1^{(l)} - f_1^{(l)} f_2^{(l-1)}) f_1 \in \mathbf{u}^-.$$

Using 1.1 (c) we see that  $x_2 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (l+1)\alpha_1 - (l-1)\alpha_2$ .

Let  $y_2 = f_1^{(l-3)} x_2 \in \mathbf{u}^-$ . According to 1.1 (c) we know that  $y_2 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (2l-2)\alpha_1 - (l-1)\alpha_2$ . It is easy to see that  $y_2 \tilde{1}_\lambda$  is maximal.

(3) Now we consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + 3\alpha_1 + 2\alpha_2),$$

$$\tilde{1}_\lambda \rightarrow t_3 = f_2^{(2)} f_1^{(3)} \tilde{1}_{\lambda+3\alpha_1+2\alpha_2}.$$

Let  $x_3 = \frac{x_{3,l-1}}{f_2^{(2)} f_1^{(3)}} \in \mathbf{u}^-$ . Using 1.1 (c) we know that  $x_3 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (2l+1)\alpha_1 - (2l-1)\alpha_2$ .

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{\mathbf{u}} f_2^{(l+2)} f_1^{(3)} \tilde{1}_{\lambda+3\alpha_1+(l+2)\alpha_2},$$

$$\tilde{1}_\lambda \rightarrow t_3 = f_2^{(l+2)} f_1^{(3)} \tilde{1}_{\lambda+3\alpha_1+(l+2)\alpha_2}.$$

Let  $y_3 = [[f_2^{(l-1)}, f_1^{(l)}], f_1^{(l)}] f_1 = \frac{x_{3,l-1}}{f_2^{(l+2)} f_1^{(3)}} \in \mathbf{u}^-$ . Using 1.1 (c) we know that  $y_3 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (2l+1)\alpha_1 - (l-1)\alpha_2$ .

(4) We consider the homomorphism:

$$\tilde{\mathbf{u}} f_1 \tilde{1}_\lambda \rightarrow \tilde{\mathbf{u}} f_2 f_1^{(3)} \tilde{1}_{\lambda+2\alpha_1+\alpha_2},$$

$$f_1 \tilde{1}_\lambda \rightarrow t_4 = f_2 f_1^{(3)} \tilde{1}_{\lambda+2\alpha_1+\alpha_2}.$$

Let  $x_4 = f_1^{(l-1)} f_2^{(l)} f_1 \in \mathbf{u}^-$ . Using 1.1 (c) we know that  $x_4 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - l\alpha_1 - l\alpha_2$ .

Let  $y_4 = \frac{x_{3,l-2}}{f_2 f_1^{(3)}} \in \mathbf{u}^-$ . Using 1.1 (c) we know that  $y_4 f_1 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - 2l\alpha_1 - (2l-2)\alpha_2$ .

(5) Now we consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l+2)\alpha_1 + (l+1)\alpha_2), \quad \tilde{1}_\lambda \rightarrow t_3 = f_1^{(l-1)} f_2^{(l+1)} f_1^{(3)} \tilde{1}_{\lambda+(l+2)\alpha_1+(l+1)\alpha_2}.$$

Let

$$x_5 = (f_2^{(l-2)} f_1^{(l)} - f_1^{(l)} f_2^{(l-2)}) \in \mathbf{u}^-.$$

Using 1.1 (c) we see that  $x_5 \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - l\alpha_1 - (l-2)\alpha_2$ .

(6) Consider the homomorphism

$$\begin{aligned}\tilde{\mathbf{u}}f_1\tilde{\mathbf{I}}_\lambda &\rightarrow \tilde{\mathbf{u}}f_2f_1^{(3)}f_2^{(2)}\tilde{\mathbf{I}}_{\lambda+2\alpha_1+3\alpha_2} \\ f_1\tilde{\mathbf{I}}_\lambda &\rightarrow f_2f_1^{(3)}f_2^{(2)}\tilde{\mathbf{I}}_{\lambda+2\alpha_1+3\alpha_2}.\end{aligned}$$

Let  $x_6 = \frac{x_{l-1,2}}{f_2f_1^{(3)}f_2^{(2)}} \in \mathbf{u}^-$ . By 1.1 (c),  $x_6f_1\tilde{\mathbf{I}}_\lambda$  is primitive and is of weight  $\lambda - 2l\alpha_1 - l\alpha_2$ .

Note that the element  $m = f_2^{(l-2)}f_1^{(2l-3)}f_2^{(l-1)}f_1\tilde{\mathbf{I}}_\lambda$  generates the unique irreducible submodule of  $\tilde{Z}(\lambda)$ . By comparing the weights of the following 6 elements with the weight of  $m$ ,

$$\begin{aligned}\frac{x_{l-1,l-1}}{f_1^{(l-1)}}\tilde{\mathbf{I}}_\lambda, & \quad \frac{x_{l-1,l-1}}{f_2^{(l)}f_1^{(l-1)}}\tilde{\mathbf{I}}_\lambda, & \quad \frac{x_{3,l-1}}{f_2^{(2)}f_1^{(3)}}\tilde{\mathbf{I}}_\lambda, \\ \frac{x_{3,l-1}}{f_2^{(l+2)}f_1^{(3)}}\tilde{\mathbf{I}}_\lambda, & \quad f_2^{(l-2)}f_1^{(l-1)}f_2^{(l)}f_1\tilde{\mathbf{I}}_\lambda, & \quad \frac{x_{l-1,2}}{f_2f_1^{(3)}f_2^{(2)}}f_1\tilde{\mathbf{I}}_\lambda,\end{aligned}$$

we see that there are no maximal elements in  $\tilde{Z}(\lambda)$  that have the same weight with any of above 6 elements.

Now we show that there are no maximal elements in  $\tilde{Z}(\lambda)$  that have the same weight with any of other 3 elements in (ii) by assuming (iii).

By (iii),  $\tilde{Z}(\lambda)$  has only one composition factor isomorphic to  $\tilde{L}(\lambda - (l+1)\alpha_1 - (l-1)\alpha_2)$ . Suppose that there is a maximal element  $m$  in  $\tilde{Z}(\lambda)$  of weight  $\lambda - (l+1)\alpha_1 - (l-1)\alpha_2$ . Then  $m$  is in  $\tilde{\mathbf{u}}y$ , here  $y = [f_2^{(l-1)}, f_1^{(l)}]f_1\tilde{\mathbf{I}}_\lambda$ . It is clear that  $\tilde{\mathbf{u}}y \subset \tilde{\mathbf{u}}^-y + \tilde{\mathbf{u}}^-f_1^{(l-3)}f_2^{(l-1)}f_1\tilde{\mathbf{I}}_\lambda$ . Thus  $m = ay + bf_1^{(3)}f_1^{(l-3)}f_2^{(l-1)}f_1\tilde{\mathbf{I}}_\lambda$  for some  $a, b$  in  $\mathbf{Q}(\xi)$ . Thus  $m = ay$ . But  $y$  is not maximal. So there are no maximal elements in  $\tilde{Z}(\lambda)$  that have the same weight with  $[f_2^{(l-1)}, f_1^{(l)}]f_1\tilde{\mathbf{I}}_\lambda$ .

Similarly, we see that there are no maximal elements in  $\tilde{Z}(\lambda)$  that have the same weight with any of  $[f_2^{(l-2)}, f_1^{(l)}]\tilde{\mathbf{I}}_\lambda$ ,  $f_1^{(l-1)}f_2^{(l)}f_1\tilde{\mathbf{I}}_\lambda$ .

(iii) Using (i), (ii) and 1.1 (b), we see that  $\tilde{Z}(\lambda)$  has the 20 composition factors. By a comparison of dimensions we know that  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

The theorem is proved.

**Theorem 3.3.** *Let  $a, b$  be integers and  $\lambda = (la + l - 2, lb + 1)$ . Then*

(i) *The following elements are maximal in  $\tilde{Z}(\lambda)$ :*

$$\begin{aligned}\tilde{\mathbf{I}}_\lambda, & \quad f_1^{(l-1)}\tilde{\mathbf{I}}_\lambda, & \quad f_2^{(2)}\tilde{\mathbf{I}}_\lambda, & \quad f_2f_1^{(l-1)}\tilde{\mathbf{I}}_\lambda, \\ f_1^{(3)}f_2^{(2)}\tilde{\mathbf{I}}_\lambda, & & \quad f_1^{(3)}f_2^{(l+1)}f_1^{(l-1)}\tilde{\mathbf{I}}_\lambda.\end{aligned}$$

$$f_2 f_1^{(3)} f_2^{(2)} \tilde{1}_\lambda, \quad f_2^{(2)} f_1^{(l+3)} f_2^{(l+1)} f_1^{(l-1)} \tilde{1}_\lambda,$$

$$\frac{f_1^{(3)} f_2}{f_1} \tilde{1}_\lambda, \quad \frac{f_2^{(2)} f_1^{(3)} f_2}{f_1} \tilde{1}_\lambda, \quad \frac{x_{3,l-2}}{f_2^{(l-2)}} \tilde{1}_\lambda.$$

(ii) The following elements are primitive elements in  $\tilde{Z}(\lambda)$  but not maximal:

$$\begin{aligned} [f_2^{(2)}, f_1^{(l)}] \tilde{1}_\lambda, \quad & \frac{x_{l-1,l-1}}{f_2^{(l-2)} f_1^{(2l-3)} f_2^{(l-1)}} \tilde{1}_\lambda, \quad \frac{f_2^{(l-1)} f_1^{(l)} f_2}{f_1} \tilde{1}_\lambda, \quad \frac{f_1 f_2^{(l-1)} f_1^{(l)} f_2}{f_1} \tilde{1}_\lambda, \\ \frac{[f_2^{(2)}, f_1^{(l)}] f_1^{(3)} f_2}{f_1} \tilde{1}_\lambda, \quad & \frac{x_{3,l-1}}{f_1 f_2^{(l-1)}} \tilde{1}_\lambda, \quad \frac{x_{3,l-1}}{f_1^{(l+1)} f_2^{(l-1)}} \tilde{1}_\lambda, \quad \frac{x_{3,l-1}}{f_1^{(2l+1)} f_2^{(l-1)}} \tilde{1}_\lambda, \\ & \frac{x_{l-1,l-1}}{f_1^{(l-3)} f_2^{(l-1)}} f_2^{(2)} \tilde{1}_\lambda. \end{aligned}$$

Moreover there are no maximal elements in  $\tilde{Z}(\lambda)$  which have the same weight with any of above 9 elements.

(iii) The maximal and primitive elements in (i-ii) provide 20 composition factors of  $\tilde{Z}(\lambda)$ , which are

$$\begin{aligned} \tilde{L}(\lambda), \quad & \tilde{L}(\lambda - (l-1)\alpha_1), \quad \tilde{L}(\lambda - 2\alpha_2), \\ \tilde{L}(\lambda - (l-1)\alpha_1 - \alpha_2), \quad & \tilde{L}(\lambda - 3\alpha_1 - 2\alpha_2) \\ \tilde{L}(\lambda - (l+2)\alpha_1 - (l+1)\alpha_2), \quad & \tilde{L}(\lambda - 3\alpha_1 - 3\alpha_2), \\ \tilde{L}(\lambda - (2l+2)\alpha_1 - (l+3)\alpha_2) \quad & \tilde{L}(\lambda - 2\alpha_1 - \alpha_2), \\ \tilde{L}(\lambda - 2\alpha_1 - 3\alpha_2), \quad & \tilde{L}(\lambda - l\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - (2l-1)\alpha_1 - l\alpha_2), \\ \tilde{L}(\lambda - (l-1)\alpha_1 - l\alpha_2), \quad & \tilde{L}(\lambda - l\alpha_1 - l\alpha_2), \quad \tilde{L}(\lambda - (l+2)\alpha_1 - 3\alpha_2), \\ \tilde{L}(\lambda - (2l+3)\alpha_1 - (l+2)\alpha_2), \quad & \tilde{L}(\lambda - (l+3)\alpha_1 - (l+2)\alpha_2), \quad \tilde{L}(\lambda - 3\alpha_1 - (l+2)\alpha_2), \\ \tilde{L}(\lambda - (2l+2)\alpha_1 - (l+1)\alpha_2), \quad & \tilde{L}(\lambda - (3l-1)\alpha_1 - 2l\alpha_2). \end{aligned}$$

Moreover,  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

*Proof:* (i) According to 1.1 (e-f), we see that the first 8 elements in (i) are maximal.

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_1), \quad \tilde{1}_\lambda \rightarrow t_1 = f_1 \tilde{1}_{\lambda + \alpha_1}.$$

Since  $f_1^{(3)} f_2$  is in  $\mathbf{u}^- f_1$ , using 1.1 (c) we see that  $\frac{f_1^{(3)} f_2}{f_1} \tilde{1}_\lambda$  and  $\frac{f_2^{(2)} f_1^{(3)} f_2}{f_1} \tilde{1}_\lambda$  are primitive elements of weights  $\lambda - 2\alpha_1 - \alpha_2$  and  $\lambda - 2\alpha_1 - 3\alpha_2$  respectively.



Now we consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + (l-2)\alpha_2), \quad \tilde{1}_\lambda \rightarrow t_2 = f_2^{(l-2)} \tilde{1}_{\lambda+(l-2)\alpha_2}.$$

Using 1.1 (c) we see that  $f_1^{(3)} f_2[f_2^{(l)}, f_1^{(l)}] f_1^{(l-1)} \tilde{1}_\lambda = \frac{x_{3,l-2}}{f_2^{(l-2)}} \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (2l+2)\alpha_1 - (l+1)\alpha_2$ .

One can check directly that the three elements are maximal.

(ii) Now we argue for (ii).

(1) Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_1), \quad \tilde{1}_\lambda \rightarrow t_1 = f_1 \tilde{1}_{\lambda+\alpha_1}.$$

Using Theorem 3.1 and 1.1 (c) we see that the first 8 elements are primitive.

(2) Now we consider the homomorphism:

$$\begin{aligned} \tilde{\mathbf{u}} f_2^{(2)} \tilde{1}_\lambda &\rightarrow \tilde{\mathbf{u}} f_1^{(l-3)} f_2^{(l-1)} \tilde{1}_{\lambda+(l-3)\alpha_1+(l-3)\alpha_2}, \\ f_2^{(2)} \tilde{1}_\lambda &\rightarrow f_1^{(l-3)} f_2^{(l-1)} \tilde{1}_{\lambda+(l-3)\alpha_1+(l-3)\alpha_2}. \end{aligned}$$

Using 1.1 (c) we know that  $\frac{x_{l-1,l-1}}{f_1^{(l-3)} f_2^{(l-1)}} f_2^{(2)} \tilde{1}_\lambda$  is a primitive element of weight  $\lambda - (3l-1)\alpha_1 - 2l\alpha_2$ .

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_1), \quad \tilde{1}_\lambda \rightarrow t_1 = f_1 \tilde{1}_{\lambda+\alpha_1}.$$

It is easy to see that all the 9 primitive elements have non-zero image. By Theorem 3.1 (ii) and 1.1 (e) we know that there are no maximal elements in  $\tilde{Z}(\lambda)$  which have the same weight with any of the 9 primitive elements.

(iii) Using (i), (ii) and 1.1 (b), we see that  $\tilde{Z}(\lambda)$  has the 20 composition factors. By a comparison of dimensions we know that  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

The theorem is proved.

**Theorem 3.4.** *Let  $a, b$  be integers and  $\lambda = (la + 2, lb + l - 2)$ . Then*

(i) *The following elements are maximal in  $\tilde{Z}(\lambda)$ :*

$$\begin{aligned} \tilde{1}_\lambda, \quad f_1^{(3)} \tilde{1}_\lambda, \quad f_2^{(l-1)} \tilde{1}_\lambda, \quad f_2^{(2)} f_1^{(3)} \tilde{1}_\lambda, \\ f_1 f_2^{(l-1)} \tilde{1}_\lambda, \quad f_1 f_2^{(2)} f_1^{(3)} \tilde{1}_\lambda, \\ f_2^{(2)} f_1^{(l+1)} f_2^{(l-1)} \tilde{1}_\lambda, \quad f_2^{(l-1)} f_1^{(2l+1)} f_2^{(l+2)} f_1^{(3)} \tilde{1}_\lambda, \end{aligned}$$

$$\frac{f_2^{(2)} f_1}{f_2} \tilde{1}_\lambda, \quad \frac{f_1^{(3)} f_2^{(2)} f_1}{f_2} \tilde{1}_\lambda.$$

(ii) The following elements are primitive elements in  $\tilde{Z}(\lambda)$  but not maximal:

$$\begin{aligned} [f_2^{(l-1)}, f_1^{(l)}] \tilde{1}_\lambda, & \quad \frac{x_{l-1, l-1}}{f_1^{(l-3)} f_2^{(l-2)} f_1^{(l-1)}} \tilde{1}_\lambda, \quad \frac{x_{3, l-2}}{f_2 f_1^{(l-1)} f_2^{(l-2)}} \tilde{1}_\lambda, \quad [f_1^{(3)}, f_2^{(l)}] \tilde{1}_\lambda, \\ \frac{x_{3, l-1}}{f_2 f_1^{(2l)} f_2^{(l-1)}} \tilde{1}_\lambda, & \quad \frac{x_{3, l-1}}{f_2 f_1^{(l)} f_2^{(l-1)}} \tilde{1}_\lambda, \quad f_1 [f_2^{(l-1)}, f_1^{(l)}] \tilde{1}_\lambda, \quad \frac{x_{l-1, 2}}{f_2 f_1^{(l-1)}} \tilde{1}_\lambda, \\ [f_2^{(2)}, f_1^{(l)}] f_1^{(3)} \tilde{1}_\lambda, & \quad \frac{x_{l-1, l-1}}{f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)}} \tilde{1}_\lambda. \end{aligned}$$

Moreover no maximal element in  $\tilde{Z}(\lambda)$  has the same weight as any of above 10 elements.

(iii) The maximal and primitive elements in (i-ii) provide 20 composition factors of  $\tilde{Z}(\lambda)$ , which are

$$\begin{aligned} & \tilde{L}(\lambda), \quad \tilde{L}(\lambda - 3\alpha_1), \quad \tilde{L}(\lambda - (l-1)\alpha_2), \\ & \tilde{L}(\lambda - 3\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - \alpha_1 - (l-1)\alpha_2) \\ & \tilde{L}(\lambda - 4\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - (l+1)\alpha_1 - (l+1)\alpha_2), \quad \tilde{L}(\lambda - (2l+4)\alpha_1 - (2l+1)\alpha_2), \\ & \tilde{L}(\lambda - \alpha_1 - \alpha_2), \quad \tilde{L}(\lambda - 4\alpha_1 - \alpha_2), \quad \tilde{L}(\lambda - l\alpha_1 - (l-1)\alpha_2), \\ & \tilde{L}(\lambda - 2l\alpha_1 - (2l-1)\alpha_2), \quad \tilde{L}(\lambda - (l+3)\alpha_1 - l\alpha_2), \quad \tilde{L}(\lambda - 3\alpha_1 - l\alpha_2), \\ & \tilde{L}(\lambda - 4\alpha_1 - (l+1)\alpha_2), \quad \tilde{L}(\lambda - (l+4)\alpha_1 - (l+1)\alpha_2), \quad \tilde{L}(\lambda - (l+1)\alpha_1 - (l-1)\alpha_2), \\ & \tilde{L}(\lambda - (l+3)\alpha_1 - (l+2)\alpha_2), \quad \tilde{L}(\lambda - (l+3)\alpha_1 - 2\alpha_2), \quad \tilde{L}(\lambda - 2l\alpha_1 - (l-1)\alpha_2). \end{aligned}$$

Moreover,  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

*Proof:* (i) According to 1.1 (e-f), we see the first 8 elements in (i) are maximal.

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_2), \quad \tilde{1}_\lambda \rightarrow t_1 = f_2 \tilde{1}_{\lambda + \alpha_2}.$$

Since  $f_2^{(2)} f_1$  is in  $\mathbf{u}^- f_2$ , using 1.1 (c) we see that  $\frac{f_2^{(2)} f_1}{f_2} \tilde{1}_\lambda$  and  $\frac{f_1^{(3)} f_2^{(2)} f_1}{f_2} \tilde{1}_\lambda$  are primitive elements of weights  $\lambda - \alpha_1 - \alpha_2$  and  $\lambda - 4\alpha_1 - \alpha_2$  respectively. One can check directly that the two elements are maximal.

(ii) Now we argue for (ii).

(1) Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_2), \quad \tilde{1}_\lambda \rightarrow t_1 = f_2 \tilde{1}_{\lambda + \alpha_2}.$$

Using Theorem 3.1 and 1.1 (c) we see that the first 9 elements are primitive.

(2) Now we consider the homomorphism:

$$\begin{aligned}\tilde{Z}(\lambda) &\rightarrow \tilde{\mathbf{u}} f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)} \tilde{\mathbf{1}}_{\lambda+(2l-4)\alpha_1+(2l-2)\alpha_2}, \\ \tilde{\mathbf{1}}_\lambda &\rightarrow f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)} \tilde{\mathbf{1}}_{\lambda+(2l-4)\alpha_1+(2l-2)\alpha_2}.\end{aligned}$$

Using 1.1 (c) we know that  $[[f_2^{(l-1)}, f_1^{(l)}], f_1^{(l)}] \tilde{\mathbf{1}}_\lambda = \frac{x_{l-1, l-1}}{f_1^{(l-3)} f_2^{(2l-2)} f_1^{(l-1)}} \tilde{\mathbf{1}}_\lambda$  is a primitive element of weight  $\lambda - 2l\alpha_1 - (l-1)\alpha_2$ .

Consider the homomorphism:

$$\tilde{Z}(\lambda) \rightarrow \tilde{Z}(\lambda + \alpha_2), \quad \tilde{\mathbf{1}}_\lambda \rightarrow t_1 = f_2 \tilde{\mathbf{1}}_{\lambda + \alpha_2}.$$

It is easy to see that all the 10 primitive elements have non-zero image. By Theorem 3.1 (ii) and 1.1 (e) we know that there are no maximal elements in  $\tilde{Z}(\lambda)$  which have the same weight with any of the 10 primitive elements.

(iii) Using (i), (ii) and 1.1 (b), we see that  $\tilde{Z}(\lambda)$  has the 20 composition factors. By a comparison of dimensions we know that  $\tilde{Z}(\lambda)$  has only the 20 composition factors.

The theorem is proved.

#### 4. WEYL MODULES FOR TYPE $B_2$

For  $\lambda = (\lambda_1, \lambda_2) \in \mathbf{Z}_+^2$  we denote by  $I_\lambda$  the left ideal of  $U_\xi$  generated by all  $e_i^{(a)}$  ( $a \geq 1$ ),  $f_i^{(a_i)}$  ( $a_i \geq \lambda_i + 1$ ),  $k_i - \xi^{\lambda_i}$ ,  $\begin{bmatrix} k_i, c \\ a \end{bmatrix} - \begin{bmatrix} \lambda_i + c \\ a \end{bmatrix}_{\xi^i}$ . The Weyl module  $V(\lambda)$  of  $U_\xi$  is defined to be  $U_\xi/I_\lambda$ , its dimension is  $(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)(\lambda_1 + 2\lambda_2 + 3)/6$ . Let  $v_\lambda$  be a nonzero element in  $V(\lambda)_\lambda$ . We can work out the maximal and primitive elements in  $V(\lambda)$  as in section 3 (cf. [X2]). We omit the results here.

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